Exercise 1.5.8

If Laplace's equation is satisfied in three dimensions, show that

$$\oiint \nabla u \cdot \hat{\boldsymbol{n}}\, dS = 0$$

for any closed surface. (*Hint*: Use the divergence theorem.) Give a physical interpretation of this result (in the context of heat flow).

Solution

Laplace's equation describes the steady-state temperature in a three-dimensional object with constant physical properties and no heat sources.

$$\nabla^2 u = 0$$

Integrate both sides over an arbitrary volume V in the object.

$$\iiint_V \nabla^2 u\, dV = \iiint_V 0\, dV$$

The right side is equal to zero.

$$\iiint_V \nabla \cdot \nabla u\, dV = 0$$

Apply the divergence theorem to the volume integral. It becomes an integral over the volume's surface S.

$$\oiint_S \nabla u \cdot \hat{\boldsymbol{n}}\, dS = 0$$

To make a physical interpretation, multiply both sides by K_0.

$$-\oiint_S (-K_0 \nabla u) \cdot \hat{\boldsymbol{n}}\, dS = 0$$

Fourier's law of conduction states that the heat flux ϕ is proportional to the temperature gradient.

$$\phi = -K_0 \nabla u$$

Substitute this result into the surface integral.

$$-\oiint_S \phi \cdot \hat{\boldsymbol{n}}\, dS = 0$$

This equation tells us that there is no net rate of heat flow through an arbitrary surface in the object.